The Pauli Exclusion Principle and the Spin-Statistics Theorem from Worldline Non-Injectivity: Exchange Phase, Rapidity, and Topological Sheet Structure

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Abstract

We derive the Pauli exclusion principle and the spin-statistics theorem from the geometry of non-injective worldlines within the TPST–DGQ framework. A timelike worldline $X^\mu(\tau)$ with Lorentz factor $\gamma > \gamma_{\text{crit}}$ intersects a constant-time hypersurface Σ_t in $N > 1$ distinct spatial points, generating a multi-sheet structure of spacetime. For a system of two identical particles, the composite worldline carries a two-particle sheet structure. We show that particle exchange is a permutation of sheet blocks that produces a topological phase η. The condition that the composite worldline be non-singular at spatial coincidence — the point where the two worldlines meet — forces $\eta = -1$ for particles whose worldlines carry an odd number of topological windings and $\eta = +1$ for particles with an even number. The winding number is identified with twice the spin of the particle via the fold-stability derivation of Planck's constant $\hbar = mc\epsilon/(2\pi\gamma_{\text{crit}})$, establishing the spin-statistics connection geometrically. The Pauli exclusion principle follows as a corollary: the amplitude for two fermions to occupy the same quantum state is identically zero because the composite worldline would require a fold with $\eta = -1$ at spatial coincidence, which cancels the amplitude exactly. We further show that the rapidity $\zeta = \operatorname{artanh}(v/c)$ is a natural measure of the inter-sheet phase difference Φ_n, so that the non-injectivity threshold $\gamma > \gamma_{\text{crit}}$ corresponds to a critical rapidity ζ_{crit} above which the multi-sheet regime is activated. The same universal cancellation identity $N(\epsilon)\cdot\epsilon^{d-2} = O(1)$ that regularises the Ryu–Takayanagi entropy, the Coulomb self-energy, and the wavefunction normalisation also regularises the two-particle exchange amplitude, providing a unified geometric origin for quantum statistics. The paper is self-contained.

1 Introduction

The Pauli exclusion principle (PEP) [1] is one of the most consequential facts of nature. It determines the structure of the periodic table, the stability of matter, the existence of white dwarfs and neutron stars, and the properties of superconductors and superfluids. Yet, within standard quantum mechanics, it is a postulate. The spin-statistics theorem [2, 3], which states that particles with half-integer spin obey Fermi-Dirac statistics while particles with integer spin obey Bose-Einstein statistics, provides a connection between spin and statistics within relativistic quantum field theory [4]. But the theorem does not derive antisymmetry from geometry: it derives it from unitarity, causality, and positivity of energy in the context of the Poincaré group — postulates that are themselves not derived from deeper principles.

The present paper shows that the PEP and the spin-statistics theorem emerge from a single geometric fact: a timelike worldline $X^\mu(\tau)$ with Lorentz factor $\gamma > \gamma_{\text{crit}}$ intersects a constant-time hypersurface Σ_t in $N > 1$ distinct spatial points. This phenomenon, called *worldline non-injectivity*, generates a multi-sheet structure of spacetime. For a system of two identical particles, the composite worldline acquires a two-particle sheet structure. Particle exchange is a permutation of sheet blocks. The topology of the exchange determines the phase $\eta = \pm 1$: the sign is fixed by the winding number of the composite worldline at the fold, which is determined by the spin of the particle.

Two additional results are established. First, the rapidity $\zeta = \operatorname{artanh}(v/c)$ is shown to be a natural measure of the inter-sheet phase difference Φ_n: the non-injectivity threshold $\gamma > \gamma_{\text{crit}}$ corresponds to a critical rapidity ζ_{crit} above which the multi-sheet structure is activated, and the speed of light c emerges as the asymptotic limit at which the phase difference between sheets diverges. Second, the same cancellation identity $N(\epsilon)\cdot\epsilon^{d-2} = O(1)$ that unifies holographic entropy, Coulomb self-energy, and wavefunction normalisation within the TPST–DGQ framework [10, 11, 12, 13] also regularises the two-particle exchange amplitude, embedding quantum statistics within the universal topological principle of worldline non-injectivity.

The paper is organised as follows. Section 2 introduces worldline non-injectivity, the Extended Lorentz Transformations (ELT), and the Ontological Identity Principle from first principles, making the paper fully self-contained. Section 3 derives Planck's constant from fold stability, following [13], since this derivation is central to the spin-statistics connection. Section 4 introduces the composite worldline for two identical particles and its sheet structure. Section 5 derives the exchange phase η from the topology of the composite worldline. Section 6 establishes the connection between winding number and spin, deriving the spin-statistics theorem. Section 7 derives the Pauli exclusion principle as a corollary. Section 8 establishes the connection between rapidity and inter-sheet phase difference. Section 9 situates the results within the universal cancellation identity. Section 10 concludes.

2 Worldline Non-Injectivity and the Multi-Sheet Structure

2.1 The injectivity assumption in standard physics

In standard special and general relativity, a physical body follows a timelike worldline $X^\mu(\tau) = (X^0(\tau), \mathbf{X}(\tau))$ parametrised by proper time τ. For any inertial observer with coordinate time $t = X^0(\tau)$, the map $\tau \mapsto t$ is implicitly assumed to be strictly monotone increasing, hence injective: each proper time corresponds to a unique coordinate time, and the body occupies exactly one spatial position at each t.

This assumption is never stated as an axiom in standard treatments. It is taken for granted. The companion paper [11] showed that it fails for worldlines with sufficiently large Lorentz factor, opening a new kinematic regime.

2.2 Definition of non-injectivity

Definition 2.1 (Non-injective worldline). *A timelike worldline $X^\mu(\tau)$ is* non-injective *with respect to the simultaneity foliation $\{\Sigma_t\}$ of an inertial observer if there exist proper times $\tau_1 \neq \tau_2$ such that:*

$$X^0(\tau_1) = X^0(\tau_2) = t^*, \qquad X^1(\tau_1) = X^1(\tau_2) = M. \tag{1}$$

The pair (t^*, M) is called a fold *of the worldline. The number of distinct proper times satisfying $X^0(\tau) = t$ for a given t is the* intersection multiplicity *$N(t)$.*

Non-injectivity arises when a body undergoes a rapid turnaround. At sufficiently high Lorentz factor, the relativistic compression of the worldline relative to the simultaneity foliation causes the outward and return trajectories to intersect the same Σ_t at the same spatial position simultaneously.

2.3 The critical Lorentz factor

The transition from injective ($N = 1$) to non-injective ($N > 1$) behaviour occurs at a critical Lorentz factor γ_{crit}. The condition for non-injectivity is:

$$\Delta\tau < \Delta\tau_{\min} = \frac{\epsilon}{\gamma_{\text{crit}}\, c}, \tag{2}$$

where ϵ is the UV cutoff and $\Delta\tau$ is the proper-time gap between two consecutive appearances. For a macroscopic back-and-forth trajectory (the Bricks Paradox), $\gamma_{\text{crit}} \approx 30$ [11]. In holographic settings, $\gamma_{\text{crit}} \sim L_{\text{AdS}}/\epsilon$.

2.4 Intersection multiplicity and UV scaling

In holographic settings, the number of intersections scales as [10]:

$$N(\epsilon) \sim \epsilon^{-(d-2)}, \tag{3}$$

where d is the number of spacetime dimensions. The universal cancellation identity follows:

$$N(\epsilon) \cdot \epsilon^{d-2} = O(1). \tag{4}$$

2.5 The Ontological Identity Principle

Definition 2.2 (Ontological Identity Principle). *The N simultaneous appearances of a physical entity at a fold of its worldline are N manifestations of a single entity. Physical properties — mass, charge, spin, and all other intrinsic quantities — are properties of the entity, not of the topological sheet, and take the same value on every sheet. Any physical operation applied to one sheet propagates coherently to all others via the continuous worldline.*

2.6 Extended Lorentz Transformations

In the non-injective regime, the standard Lorentz boost is replaced by N Extended Lorentz Transformations (ELT), one per sheet [11]. For a boost along x^1 with velocity v and Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2}$:

$$t'_n = \gamma\left(t - \frac{vx}{c^2}\right), \tag{5}$$

$$x'_n = \gamma(x - vt) + \Phi_n, \tag{6}$$

where the *topological phase offset* of the n-th sheet is:

$$\Phi_n = \gamma^2 v(\tau_n - \tau_1). \tag{7}$$

For $N = 1$, $\Phi_1 = 0$ and the ELT reduces to the standard Lorentz boost. The N sheets are connected by the continuous worldline, so the phase offsets Φ_n are not free parameters but are determined by the worldline geometry.

3 Planck's Constant from Fold Stability

We summarise the derivation of $\hbar$ from [13], since it is central to the spin-statistics connection established in Section 6.

3.1 Minimum fold separation

The worldline folds back on Σ_t at consecutive intersection points $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$. Two folds closer than the UV cutoff ϵ cannot be resolved as distinct intersections. The minimum spatial separation between two stable consecutive folds is therefore:

$$|\mathbf{X}_{i+1}(t) - \mathbf{X}_i(t)|_{\min} = \epsilon. \tag{8}$$

3.2 Minimum proper-time gap

For a worldline near γ_{crit} with $v \approx c$:

$$\Delta\tau_{\min} = \frac{\epsilon}{\gamma_{\text{crit}}\, c}. \tag{9}$$

3.3 Stability condition

Two consecutive folds are stable if they do not merge. Merging is prevented when the inter-sheet electromagnetic fields (Maxwell Topological Emergence Identity of [10]) produce destructive interference over one complete cycle. The minimum phase difference for this to occur is:

$$\Delta\Phi_{\min} = 2\pi. \tag{10}$$

3.4 Derivation of $\hbar$

The minimum action between two stable folds is:

$$S_{\min} = mc^2\, \Delta\tau_{\min} = \frac{mc\,\epsilon}{\gamma_{\text{crit}}}. \tag{11}$$

The quantum of action per radian of phase is:

$$\boxed{\hbar = \frac{S_{\min}}{\Delta\Phi_{\min}} = \frac{mc\,\epsilon}{2\pi\,\gamma_{\text{crit}}}.} \tag{12}$$

Inverting:

$$\epsilon = \frac{2\pi\hbar\,\gamma_{\text{crit}}}{mc} = \bar{\lambda}_C \cdot 2\pi\gamma_{\text{crit}}, \tag{13}$$

where $\bar{\lambda}_C = \hbar/(mc)$ is the reduced Compton wavelength. This confirms that the UV cutoff at which folds become resolvable is the Lorentz-boosted Compton wavelength of the particle [13].

3.5 Winding number and phase quantisation

The stability condition $\Delta\Phi_{\min} = 2\pi$ implies that the minimum stable phase accumulated between two consecutive folds is one complete oscillation. The total phase accumulated over w complete cycles is:

$$\Phi^{(w)} = 2\pi w, \qquad w \in \mathbb{Z}. \tag{14}$$

The integer w is the *winding number* of the worldline around the fold structure. For a worldline with winding number w, the minimum action per fold cycle is:

$$S^{(w)}_{\min} = w \cdot S_{\min} = \frac{wmc\,\epsilon}{\gamma_{\text{crit}}}, \tag{15}$$

giving a generalised Planck constant:

$$\hbar^{(w)} = \frac{S^{(w)}_{\min}}{2\pi w} = \hbar. \tag{16}$$

The winding number cancels in the definition of $\hbar$ itself, but it leaves a physical signature in the exchange phase, as shown in Section 5.

4 The Composite Worldline for Two Identical Particles

4.1 Single-particle sheet structure

For a single particle with worldline $X_1^\mu(\tau)$ in the non-injective regime, the intersection multiplicity $N_1(\epsilon) \sim \epsilon^{-(d-2)}$ generates N_1 sheets, labelled by the index $i \in \{1, \ldots, N_1\}$. The intersection points on Σ_t are $\{\mathbf{X}_{1,i}(t)\}$. By the Ontological Identity Principle, all sheets carry the same intrinsic properties: mass m, charge q, and spin s.

4.2 Two-particle composite worldline

For two identical particles with worldlines $X_1^\mu(\tau_1)$ and $X_2^\mu(\tau_2)$, define the *composite worldline* in the two-particle configuration space:

$$\mathcal{X}^\mu(\tau_1, \tau_2) := \big(X_1^\mu(\tau_1),\, X_2^\mu(\tau_2)\big). \tag{17}$$

The composite worldline lives in the product space $\mathcal{M} \times \mathcal{M}$, where $\mathcal{M}$ is the single-particle spacetime.

In the non-injective regime, each single-particle worldline generates N_1 and N_2 sheets respectively. The composite worldline generates $N_1 \times N_2$ sheet pairs. However, the particles are identical by assumption.

Definition 4.1 (Identical particles in the multi-sheet framework)**.** *Two particles are* identical *if their worldlines carry the same intrinsic quantum numbers* (m, q, s) *and their sheet structures are isomorphic:* $N_1 = N_2 = N$ *and the topological phase offsets satisfy* $\Phi_i^{(1)} = \Phi_i^{(2)}$ *for all* i*.*

Under this definition, the $N \times N$ sheet pairs of the composite worldline reduce to N independent pairs when the particles are identical, because indistinguishability identifies sheet i of particle 1 with sheet i of particle 2.

4.3 Physical states of the composite worldline

The physical states of the composite worldline are characterised by the relative configuration of the two particles across the N sheets. At each coordinate time t, the two-particle state is:

$$\left\{ \left(\mathbf{X}_{1,i}(t),\, \mathbf{X}_{2,j}(t) \right) \right\}_{i,j=1}^{N}. \tag{18}$$

The indistinguishability of the particles imposes that the physical state must be invariant (up to a phase) under the exchange of particle labels $1 \leftrightarrow 2$.

5 Exchange as a Permutation of Sheet Blocks

5.1 The exchange operator

The exchange operator P_{12} acts on the two-particle state (18) by swapping the two worldlines:

$$P_{12} : \left(X_1^\mu(\tau_1),\, X_2^\mu(\tau_2) \right) \longmapsto \left(X_2^\mu(\tau_2),\, X_1^\mu(\tau_1) \right). \tag{19}$$

In the multi-sheet language, P_{12} permutes the blocks of N sheets associated with particle 1 and the N sheets associated with particle 2.

5.2 Topological constraint on the exchange

Since $P_{12}^2 = \mathbb{I}$ (two exchanges return to the original state), the exchange operator satisfies:

$$P_{12}^2 = \mathbb{I} \quad \Rightarrow \quad \eta^2 = 1 \quad \Rightarrow \quad \eta = \pm 1, \tag{20}$$

where η is the phase factor produced by a single exchange. The sign $\eta = +1$ corresponds to bosons (symmetric states) and $\eta = -1$ to fermions (antisymmetric states).

5.3 The exchange path in configuration space

In the multi-sheet framework, the exchange of two identical particles is not a point operation but a *path* in the two-particle configuration space. Specifically, the exchange corresponds to moving particle 1 along a path from $\mathbf{X}_{1,i}(t)$ to $\mathbf{X}_{2,j}(t)$ while simultaneously moving particle 2 from $\mathbf{X}_{2,j}(t)$ to $\mathbf{X}_{1,i}(t)$.

In three spatial dimensions, the configuration space of two identical particles is $(\mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta)/\mathbb{Z}_2$, where Δ is the diagonal (coincident positions) and $\mathbb{Z}_2$ is the exchange group. The fundamental group of this space is $\pi_1 = \mathbb{Z}_2$, which has exactly two elements: the trivial path (no exchange) and the non-trivial path (one exchange) [5]. This is why $\eta = \pm 1$ exhausts all possibilities in three dimensions.

5.4 The exchange phase from the winding number

In the non-injective regime, the exchange path passes through the fold structure of the composite worldline. The topological phase accumulated along the exchange path is:

Theorem 5.1 (Exchange Phase from Winding Number)**.** *Let the single-particle worldline have winding number w around its fold structure, as defined by* (14). *Under a single exchange of two identical particles, the composite worldline acquires a topological phase:*

$$\eta = e^{i\pi w} = (-1)^w. \tag{21}$$

Proof. The exchange path in the two-particle configuration space begins at $(\mathbf{X}_{1,i}, \mathbf{X}_{2,j})$ and ends at $(\mathbf{X}_{2,j}, \mathbf{X}_{1,i})$. Along this path, particle 1 traces a half-loop in configuration space: it moves from one intersection point to the position previously occupied by particle 2.

In the non-injective regime, the worldline of each particle winds w times around the fold before returning to its initial sheet. A single exchange corresponds to traversing *half* the total winding cycle (because two exchanges return to the starting configuration). The phase accumulated over half the winding cycle is:

$$\Phi_{\text{exchange}} = \frac{1}{2} \cdot 2\pi w = \pi w. \tag{22}$$

The topological phase factor is therefore $e^{i\pi w} = (-1)^w$. □

Remark 5.2. *The factor of* $1/2$ *in* (22) *arises from the same geometric fact that underlies the spin-*$1/2$ *property of fermions: a rotation by* 2π *in the spatial rotation group* $SO(3)$ *corresponds to a phase of* $(-1)^{2s}$ *in the spinor representation, where* s *is the spin. A single exchange in the configuration space of two identical particles is topologically equivalent to a* 2π *rotation of one particle relative to the other [6, 5].*

6 Spin-Statistics Connection

6.1 Winding number and spin

We now establish the connection between the winding number w of the worldline and the spin s of the particle.

In the fold-stability derivation of $\hbar$ (Section 3), the minimum phase per fold cycle is $\Delta\Phi_{\text{min}} = 2\pi$, corresponding to one complete oscillation of the inter-sheet electromagnetic field. This is the topological condition for a stable fold.

The spin of a particle is the intrinsic angular momentum of its worldline under spatial rotations. In the non-injective framework, the worldline folds back on itself. The fold structure under a rotation by angle θ accumulates a phase:

$$\Phi_{\text{rot}}(\theta) = s \cdot \theta, \tag{23}$$

where s is the spin quantum number. Under a full rotation $\theta = 2\pi$:

$$\Phi_{\text{rot}}(2\pi) = 2\pi s. \tag{24}$$

For the fold to be stable, this phase must be a multiple of the minimum stable phase $\Delta\Phi_{\text{min}} = 2\pi$:

$$2\pi s = w \cdot 2\pi \quad \Rightarrow \quad s = w \cdot \frac{1}{1} = w. \tag{25}$$

However, the spin is not required to be an integer by the fold stability condition alone. The precise relationship is established by the following argument.

The worldline fold is a topological structure in spacetime, and the stability condition requires the phase to be single-valued under $\theta = 4\pi$ (since the physical state must return to itself after two full rotations, by the 2π periodicity of the rotation group $SO(3)$ in three dimensions):

$$\Phi_{\text{rot}}(4\pi) = 4\pi s \equiv 0 \pmod{2\pi}. \tag{26}$$

This requires $2s \in \mathbb{Z}$, i.e. s is a half-integer or integer. The winding number of the fold is then:

$$w = 2s. \tag{27}$$

Theorem 6.1 (Spin-Statistics from Non-Injectivity)**.** *Let $w = 2s$ be the winding number of a particle worldline with spin s. Under a single exchange of two identical particles, the composite worldline acquires the phase:*

$$\eta = (-1)^w = (-1)^{2s}. \tag{28}$$

For half-integer spin ($s = 1/2, 3/2, \ldots$): w is odd, $\eta = -1$ (fermions, antisymmetric). For integer spin ($s = 0, 1, 2, \ldots$): w is even, $\eta = +1$ (bosons, symmetric). This is the spin-statistics theorem.

Proof. By Theorem 5.1, $\eta = (-1)^w$. By (27), $w = 2s$. Therefore $\eta = (-1)^{2s}$. For half-integer s, $2s$ is odd, giving $\eta = -1$. For integer s, $2s$ is even, giving $\eta = +1$. □

Remark 6.2 (Comparison with standard derivation)**.** *The standard spin-statistics theorem [2, 4] derives the connection from unitarity, causality, and positivity of energy in relativistic quantum field theory. Theorem 6.1 derives the same result from the winding number of the worldline fold structure — a purely geometric quantity that does not presuppose the operator formalism of QFT.*

7 The Pauli Exclusion Principle

Theorem 7.1 (Pauli Exclusion Principle from Non-Injectivity)**.** *Let two identical fermions (particles with half-integer spin, $\eta = -1$) attempt to occupy the same quantum state, i.e. let their worldlines coincide spatially: $\mathbf{X}_1(t) = \mathbf{X}_2(t)$ for some t. Then the amplitude for this configuration is identically zero.*

Proof. **Step 1: Spatial coincidence and the singularity condition.** Consider the two-particle amplitude $\mathcal{A}(\mathbf{x}_1, \mathbf{x}_2)$ for finding particle 1 at $\mathbf{x}_1$ and particle 2 at $\mathbf{x}_2$ at coordinate time t. At spatial coincidence $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$, the composite worldline has a fold: both particles appear at the same spatial point on the same sheet.

Step 2: Exchange symmetry at coincidence. At spatial coincidence, the two-particle state is mapped to itself under exchange P_{12}, because swapping two particles at the same position produces the same configuration:

$$P_{12}\, \mathcal{A}(\mathbf{x}, \mathbf{x}) = \eta\, \mathcal{A}(\mathbf{x}, \mathbf{x}). \tag{29}$$

But also:

$$P_{12}\, \mathcal{A}(\mathbf{x}, \mathbf{x}) = \mathcal{A}(\mathbf{x}, \mathbf{x}), \tag{30}$$

since swapping identical particles at the same point leaves the physical configuration unchanged.

Step 3: Vanishing of the amplitude. From (29) and (30):

$$\mathcal{A}(\mathbf{x}, \mathbf{x}) = \eta\, \mathcal{A}(\mathbf{x}, \mathbf{x}). \tag{31}$$

For fermions, $\eta = -1$, so:

$$\mathcal{A}(\mathbf{x}, \mathbf{x}) = -\mathcal{A}(\mathbf{x}, \mathbf{x}) \quad \Rightarrow \quad \mathcal{A}(\mathbf{x}, \mathbf{x}) = 0. \tag{32}$$

The amplitude is identically zero.

Step 4: Extension to identical quantum states. The argument extends from spatial coincidence to full quantum state coincidence. Two identical fermions in the same quantum state (n, ℓ, m_ℓ, m_s) have the same spatial wavefunction and the same spin projection. Their composite wavefunction $\Psi(\mathbf{x}_1, \sigma_1; \mathbf{x}_2, \sigma_2)$ must satisfy:

$$\Psi(\mathbf{x}_1, \sigma_1; \mathbf{x}_2, \sigma_2) = -\Psi(\mathbf{x}_2, \sigma_2; \mathbf{x}_1, \sigma_1). \tag{33}$$

If $\mathbf{x}_1 = \mathbf{x}_2$ and $\sigma_1 = \sigma_2$ (same state):

$$\Psi(\mathbf{x}, \sigma; \mathbf{x}, \sigma) = -\Psi(\mathbf{x}, \sigma; \mathbf{x}, \sigma) = 0. \tag{34}$$

□

Corollary 7.2 (Stability of Matter)**.** *The Pauli exclusion principle, derived here from the topology of the composite worldline, implies the stability of matter: electrons in an atom cannot all collapse to the ground state, because identical fermions cannot occupy the same quantum state. This is a consequence of worldline non-injectivity, not a postulate.*

8 Rapidity as a Measure of Inter-Sheet Phase

8.1 Definition of rapidity

In special relativity, the rapidity is:

$$\zeta = \operatorname{artanh}\left(\frac{v}{c}\right) = \frac{1}{2} \ln\left(\frac{1 + v/c}{1 - v/c}\right), \tag{35}$$

with $\gamma = \cosh\zeta$ and $v/c = \tanh\zeta$. Rapidities are additive under composition of boosts: $\zeta_{12} = \zeta_1 + \zeta_2$. As $v \to c$, $\zeta \to \infty$.

8.2 Non-injectivity threshold in rapidity language

The non-injectivity condition $\gamma > \gamma_{\mathrm{crit}}$ is equivalent to:

$$\zeta > \zeta_{\mathrm{crit}} := \operatorname{arccosh}(\gamma_{\mathrm{crit}}). \tag{36}$$

For $\zeta < \zeta_{\mathrm{crit}}$: the worldline is injective ($N = 1$), standard Lorentz boost, no multi-sheet structure. For $\zeta > \zeta_{\mathrm{crit}}$: the worldline is non-injective ($N > 1$), ELT replaces the standard boost, multi-sheet structure activated.

Theorem 8.1 (Rapidity as Phase Measure)**.** *The inter-sheet phase difference* $\Phi_n = \gamma^2 v(\tau_n - \tau_1)$ *is a monotone increasing function of the rapidity* ζ*. In the non-relativistic limit* $v \ll c$*:*

$$\Phi_n \approx v(\tau_n - \tau_1) \approx c\ \tanh\zeta \cdot \Delta\tau_n \approx c\,\zeta \cdot \Delta\tau_n, \tag{37}$$

where $\Delta\tau_n = \tau_n - \tau_1$*. In the relativistic regime:*

$$\Phi_n = \gamma^2 v\,\Delta\tau_n = \frac{c\sinh\zeta\cosh\zeta}{1}\,\Delta\tau_n = \frac{c}{2}\sinh(2\zeta)\,\Delta\tau_n. \tag{38}$$

The inter-sheet phase diverges as $\zeta \to \infty$ *(i.e.* $v \to c$*).*

Proof. From $v = c\tanh\zeta$ and $\gamma = \cosh\zeta$: $\gamma^2 v = c\cosh^2\zeta\tanh\zeta = c\cosh\zeta\sinh\zeta = (c/2)\sinh(2\zeta)$. Substituting into (7): $\Phi_n = (c/2)\sinh(2\zeta)\,\Delta\tau_n$. For $v \ll c$, $\zeta \approx v/c \ll 1$ and $\sinh(2\zeta) \approx 2\zeta$, giving $\Phi_n \approx c\zeta\,\Delta\tau_n$. □

8.3 Physical interpretation

Theorem 8.1 shows that the rapidity is not merely a mathematical convenience for composing boosts: it is a direct measure of the inter-sheet phase offset of the non-injective worldline.

Corollary 8.2 (Speed of Light as Phase Divergence)**.** *The speed of light c is the asymptotic limit at which the inter-sheet phase difference* $|\Phi_n|$ *diverges for any non-zero proper-time gap* $\Delta\tau_n > 0$*. No physical system can reach* $v = c$ *because the phase divergence would require infinite action between consecutive folds, which is incompatible with the finite stability condition* $S_{\min} = mc\epsilon/\gamma_{\text{crit}}$ *of Section 3.*

Proof. As $v \to c$, $\zeta \to \infty$, and $\Phi_n = (c/2)\sinh(2\zeta)\Delta\tau_n \to \infty$ for any $\Delta\tau_n > 0$. The action per fold $S = mc^2\,\Delta\tau_n$ is finite, so the phase per unit action $\Phi/S = (c/2)\sinh(2\zeta)\Delta\tau_n/(mc^2\Delta\tau_n) = \sinh(2\zeta)/(2mc)$ diverges. This exceeds the finite stability budget $\hbar^{-1}$ for any massive particle, making $v = c$ unreachable. □

8.4 Rapidity and the non-injectivity degree

The number of sheets $N(\epsilon)$ and the rapidity ζ are related through the following chain. By (3), $N \sim \epsilon^{-(d-2)}$. By (13), $\epsilon = 2\pi\bar{\lambda}_C\gamma_{\text{crit}}$. The non-injectivity threshold $\gamma_{\text{crit}} = \cosh\zeta_{\text{crit}}$ connects the two. For a worldline at rapidity $\zeta > \zeta_{\text{crit}}$, the number of active sheets is:

$$N(\zeta) \sim \left(\frac{\cosh\zeta_{\text{crit}}}{\cosh\zeta}\right)^{-(d-2)} \cdot N_{\text{crit}}, \tag{39}$$

where $N_{\text{crit}} = N(\epsilon)|_{\zeta=\zeta_{\text{crit}}}$ is the sheet count at threshold. As ζ increases above ζ_{crit}, $N(\zeta)$ increases, activating more sheets and deepening the non-injective structure.

9 Universal Cancellation and Quantum Statistics

The cancellation identity $N(\epsilon) \cdot \epsilon^{d-2} = O(1)$ now operates at seven levels:

Level	UV-divergent object	Regularised result
Holography	RT area $\sim \epsilon^{-(d-2)}$	$S_{\mathrm{DG}} = O(1)$
Classical EM	Coulomb energy $\sim \epsilon^{-(d-2)}$	$\langle \mathcal{E} \rangle = O(1)$
Quantum mechanics	Intersection density $\sim \epsilon^{-(d-2)}$	$\lvert\psi\rvert^2 = O(1)$
Thermodynamics	Single-sheet entropy	$S_{\mathrm{top}} \geq 0$
EM fields	$\delta F^{(n)} \sim \epsilon^{d-2}$	$\langle F \rangle = F^{\mathrm{std}}$
Gravity	$\Lambda_{\mathrm{bare}} \sim \epsilon^{-2}$	$\Lambda_{\mathrm{obs}} = \Lambda_0/N_0 = O(1)$
Statistics	Exchange amplitude at coincidence $\sim \epsilon^{-(d-2)}$	$\mathcal{A}(\mathbf{x}, \mathbf{x}) = 0$ (fermions) or $O(1)$ (bosons)

The regularisation of the exchange amplitude at spatial coincidence deserves comment. For two fermions approaching $\mathbf{x}_1 \to \mathbf{x}_2 = \mathbf{x}$, each single-particle intersection density scales as $\rho_i(\mathbf{x}, t) \sim \epsilon^{-(d-2)}$ (UV-divergent in the naive single-sheet theory). The two-particle exchange amplitude would naively be of order $\epsilon^{-2(d-2)}$. However, the topological cancellation $\eta = -1$ forces the total amplitude to zero, independently of the UV behaviour. For bosons ($\eta = +1$), the two-particle amplitude at coincidence is of order $O(1)$ by the standard topological averaging (4). This is the statistical analogue of the UV cancellation in holography and electrodynamics: the worldline topology — not an external regulator — determines the physical result.

Remark 9.1 (Dark matter and dark energy)**.** *The multi-sheet framework has potential implications for cosmology. Spatial variations of $N(x, t)$ on galactic scales could contribute an effective mass density without electromagnetic interactions, providing a geometric origin for dark matter. Temporal evolution of $N(t)$ on cosmological scales is connected to the observed cosmological constant $\Lambda_{\mathrm{obs}} = \Lambda_{\mathrm{bare}}/N$, providing a geometric origin for dark energy. A systematic derivation of these effects from the TPST–DGQ framework, with comparison to observational data, is left to future work.*

10 Conclusions

We have derived the Pauli exclusion principle and the spin-statistics theorem from the geometry of non-injective worldlines.

Exchange phase. For two identical particles, the composite worldline generates a two-particle sheet structure. Particle exchange is a permutation of sheet blocks. The topological phase produced by a single exchange is $\eta = (-1)^w$, where w is the winding number of the worldline around its fold structure (Theorem 5.1).

Spin-statistics theorem. The winding number is related to the spin by $w = 2s$, established by the requirement that the physical state be single-valued under a 4π rotation

(Theorem 6.1). Therefore $\eta = (-1)^{2s}$: half-integer spin gives $\eta = -1$ (fermions), integer spin gives $\eta = +1$ (bosons).

Pauli exclusion principle. The amplitude for two fermions at spatial coincidence satisfies $\mathcal{A} = \eta\mathcal{A}$ with $\eta = -1$, forcing $\mathcal{A} = 0$. No two fermions can occupy the same quantum state (Theorem 7.1). This is a theorem of the geometry of non-injective worldlines, not a postulate.

Rapidity and inter-sheet phase. The rapidity ζ is a monotone function of the inter-sheet phase difference $\Phi_n = (c/2)\sinh(2\zeta)\Delta\tau_n$ (Theorem 8.1). The non-injectivity threshold $\gamma > \gamma_{\text{crit}}$ corresponds to $\zeta > \zeta_{\text{crit}}$. The speed of light c is the asymptotic limit at which the phase diverges, making $v = c$ unreachable for massive particles (Corollary 8.2).

Universal cancellation. The same identity $N(\epsilon) \cdot \epsilon^{d-2} = O(1)$ that regularises holographic entropy, Coulomb self-energy, wavefunction normalisation, electromagnetic fields, and the cosmological constant also determines the exchange amplitude: zero for fermions, finite for bosons (Section 9).

The central identity remains:

$$\text{Non-injectivity} \iff \text{Finite physics at every level.} \tag{40}$$

Declarations

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